A characterization of n-player strongly monotone scheduling mechanisms

Submission $\#705$

Abstract

Our work deals with the important problem of globally characterizing truthful mechanisms where players have multi-parameter valuations like scheduling unrelated machines or combinatorial auctions. Very few mechanisms are known for these multi-parameter settings and the question is: Can we prove that no other truthful mechanisms exist?

We characterize truthful mechanisms for n players and 2 tasks for multiple settings: scheduling urelated machines, combinatorial auctions where the players have additive or subadditive or submodular valuations and all items have to be allocated: A truthful mechanism is either task-independet or a player-grouping minimizer, a new class of mechanisms we discover and which generalizes affine minimizers. We assume decisiveness, strong monotonicity and that the boundaries in which the mechanism partitions the input space are continuous functions.

1 Introduction

Using the power of *crowdsourcing* and *cloud computing* to compute complicated tasks that consist of multiple sub-tasks is a major challenge in multi-agent systems. When assigning the tasks to different agents/cloud providers, we have to provide the agents with the right incentives to truthfully report the times they need to complete the tasks, and execute the tasks that are assigned to them.

Combinatorial auctions constitute another important class of problems. Here multiple items are auctioned simultaneously, and we need to motivate the agents to report their true valuations for the items. Can we characterize all allocation algorithms that are truthful for these two settings?

Some examples of truthful mechanisms are Dictatorships, the Vickrey Auction, which maximizes the welfare of the players, Affine maximizers(/minimizers), which maximize a weighted welfare of the players (there is an additive weight for each outcome and a positive weight for each player), Threshold mechanisms, which allocate each item "almost independently". Some of the most famous results in microeconomics like the Gibbard-Sattherwhaite theorem, Arrow's theorem [Kenneth, 1951] and the Roberts' theorem [Roberts, 1979] provide characterizations of truthful mechanisms.

Fifteen years ago Nisan and Ronen posed their famous – still open– question about the approximability of the optimum makespan in unrelated machine scheduling by *truthful scheduling mechanisms* [Nisan and Ronen, 2001]. In this strategic version of the unrelated machines problem, we are given n machines and m tasks, and the machines are owned by selfish agents, each of them holding the vector $t_i = (t_{ij})_{j=1}^{m}$ of running times (costs) on his machine i as private information. A scheduling mechanism consists of an *allocation algorithm*, and a *payment scheme* $(p_1, \ldots, p_n)$. Having received the *bid* vectors t'_i for the costs from the respective agents, the matrix t' is used as input of the allocation algorithm, and the payments to each agent are calculated according to the payment scheme. The *utility* of player i is then $p_i - cost_i$, where $cost_i$ is defined as the total running time of the received jobs on machine i, i.e., the finish time of the machine. Note that for each player, the costs incurred from the different tasks are additive. A similar problem to unrelated scheduling is that of *combinatorial auctions (CAs)* with *additive* bidder valuations: we get a model equivalent to additive CAs by assuming *negative* values of t_{ij}, and leaving the scheduling model unchanged otherwise.

We are interested in *truthful* mechanisms, where bidding $t'_i = t_i$ is a dominant strategy for every agent. It is well known that *weak monotonicity (WMON)* of the allocation function, is necessary and sufficient for truthful implementability(see [Saks and Yu, 2005; Archer and Kleinberg, 2008]). Thus, the problem boils down to searching for monotone allocation algorithms for unrelated scheduling.

While monotonicity characterized truthful mechanisms well in the (single-parameter) *related machines* case ([Epstein *et al.*, 2013] and references therein), it is much more difficult to directly exploit it in multi-parameter settings like unrelated scheduling. A different approach is to improve our understanding by investigating the *global structure* of WMON allocations. To this end we strive for *global, closed form* characterizations. Even though it seems extremely hard to provide a complete characterization for the original problem, attempts to characterize restricted, in some way purified classes of WMON mechanisms prove to be very useful: they develop insight, while new types of allocations might be discovered along the way. The question we pose is: *Do we encounter very complex mechanisms (convincing that any attempt for a complete characterization is doomed to failure); or do monotone allocations remain managable (at least) in the restricted classes?*

In this paper we assume *strong monotonicity (SMON)*, a condition that parallels Arrow's Independence of Irrelevant Alternatives (IIA) condition, and which is the strict version of the WMON property (see Definition 3). We already knew from[Mu'alem and Schapira, 2007] that SMON mechanims can only approximate the makespan by factor $\min(m, n)$.

We *completely characterize* SMON mechanisms for two tasks, and at the same time identify a new class of monotone allocations. Our characterization also implies a characterization of essentialy all superdomains of additive valuations (like submodular, subadditive or superadditive valuations), by applying the black box reduction introduced in [Vidali, 2011].

Related work. Characterizations by *weak monotonicity* [Myerson, 1981; Saks and Yu, 2005; Gui *et al.*, 2005; Archer and Kleinberg, 2008; Frongillo and Kash, 2012], and *cycle monotonicity* [Rochet, 1987] describe truthfully implementable allocations in a *local* fashion.

Complete characterizations, of implementable allocations describe them in a *global* fashion. The most important result of this type is due to Roberts who showed that for *unrestricted* domains the only implementable social choice rules are affine maximizers [Roberts, 1979], a generalization of VCG mechanisms. However, the requirement of unrestricted valuations does not apply to most of the realistic setups with richer structure. Characterizations for domains with high economic importance like combinatorial auctions or the scheduling domain seem very hard to obtain, even with additional restrictions. Lavi et al. [Lavi *et al.*, 2003] showed that assuming a property analogous to SMON (IIA), the only truthful mechanisms in *order based domains* are so-called "almost-" affine maximizers. Dobzinski and Nisan characterize the only *scalable* multi-unit auctions for 2 items and 2 players with better than 2-approximation of the welfare, and term these *triage auctions* [Dobzinski and Nisan, 2011]. Ashlagi and Serizawa characterize truthful, individually rational multi-unit auctions, where each bidder receives at most one item. They assume *anonymity in welfare*, and show that the only allocation rule is the VCG allocation [Ashlagi and Serizawa, 2012].

The characterizations we know for the case of n players, either concern domains where SMON can be assumed without loss of generality (unrestricted domain or public projects[Roberts, 1979; Papadimitriou *et al.*, 2008]) or put additional restrictions: [Lavi *et al.*, 2003; Dobzinski and Sundararajan, 2008; Lavi *et al.*, 2009] assume SMON (and decisiveness), [Dobzinski and Sundararajan, 2008; Dobzinski and Nisan, 2011] assume continuity and scalability or [Christodoulou and Kovács, 2011] substitute truthfulness with envy-freeness. All these characterizations, even under the extra assumptions are complicated and of huge value. The only complete characterization results for *additive* bidders in the multi-parameter setting are for two players [Dobzinski and Sundararajan, 2008; Christodoulou *et al.*, 2008]. These characterizations involve *affine minimizers*, or *threshold* mechanisms, or a combination of these.[1] We define these allocation rules next.

Definition 1 (affine minimizer). *An allocation function A is an* affine minimizer *if there exist positive multiplicative constants λ_i for each player i, and additive constants c_a one for each allocation a, such that for every input matrix $t = \{t_{ij}\}_{n\times m}$ the allocation $A(t) = \{a_{ij}\}_{n\times m}$ minimizes $\sum_{i=1}^{n}\sum_{j=1}^{m} \lambda_i \cdot a_{ij} \cdot t_{ij} + c_a$ (where a_{ij} is 1 if player i gets task j and 0 otherwise).*

Threshold allocations are exactly those that admit additive payment functions over the received tasks/items [Vidali, 2009]. Restricted to SMON mechanisms, they coincide (assuming proper tie-breaking) with *task-independent* mechanisms, that allocate each task by an arbitrary monotone single item allocation. Single item mechanisms were characterized as *virtual utility maximizers* in [Mishra and Quadir, 2012].

Our contribution. **1.** We started trying to extend the characterization for 2 players [Christodoulou *et al.*, 2008] identify a so far unknown monotone allocation rule generalizing affine minimizers, that we call *player-grouping minimizer*. A player-grouping minimizer partitions the set of players, and always allocates all tasks within one subset of the partition (called *group*) by some affine minimizer. Between any two groups of players, preferences are decided by minimizing arbitrary fixed increasing functions of the objective values of the groups:

Definition 2 (player-grouping minimizer). *Let $\{N_g\}_{g=1}^{r}$ be a partition of the set of players into r* groups, *with at least two players in each group. For each $1 \le g \le r$ let $\Phi_g : (-\infty, C_g) \to \mathbb{R}$ be an increasing continuous bijection[2] and A_g be an affine minimizer over the players of group g. Within each group the affine minimizer A_g decides, which players (would) receive the tasks in that group. For given bids t_i of the players in group g, let Opt_g denote the objective value of A_g.[3] Group s receives all the tasks, allocated according to A_s, if $\Phi_s(Opt_s) = \min_g \Phi_g(Opt_g)$ (assuming some consistent tie-breaking rule).*

Example 1. *The allocation for $n = 4$ and $m = 2$, that gives the tasks to the players who provide the minimum of the expressions $t_{11} + t_{12}$, $t_{11} + 5t_{22}$, $5t_{21} + t_{12}$, $5t_{21} + 5t_{22}$, and $(min\{t_{31} + t_{32} + 3, t_{31} + t_{42}, t_{41} + t_{32} - 1, t_{41} + t_{42}\})^3$*

is a grouping minimizer. If ties are broken by a fixed order of these eight possible allocations, then it is an SMON grouping minimizer.

2. We characterize SMON mechanisms for two tasks or items as either task-independent mechanisms or player-grouping minimizers:

[1] In CAs, *affine maximizers* become *affine minimizers*.

[2] The $C_g \in \mathbb{R} \cup \{+\infty\}$ with C_g being $+\infty$ for at least one g; this is needed for the tasks to be always allocated.

[3] $Opt_g = \min_{a^g} \sum_i \sum_j \lambda_i t_{ij} a^g_{ij} + c_{a^g}$, where allocations a^g give all tasks to group g.

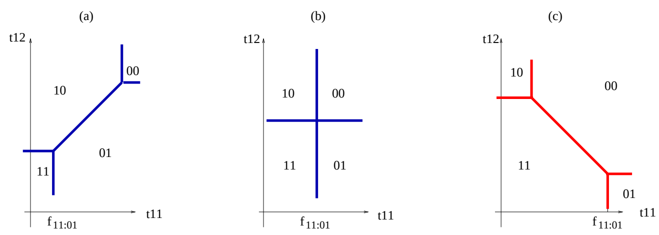

Figure 1: The allocations to a single player depending on his own 2-dimensional bid vector, partitions the bid-space according to one of these shapes.

Theorem 1. *Every continuous decisive SMON mechanism for allocating two tasks or items, with additive bidder valuations and $t_{ij} \in \mathbb{R}$, is either a task-independent mechanism, or a grouping minimizer (if the grouping minimizer is onto*[4]*, then it is an affine minimizer).*

Since monotone allocations for two players are essentially SMON (with an appropriate tie-breaking rule), for two tasks our result generalizes the 2-player characterization in [Christodoulou *et al.*, 2008]. Moreover, grouping minimizers are very similar to *virtual utility maximizers* for a single item [Mishra and Quadir, 2012].

3. We derive an elegant lemma (Lemma 2) that turns out to be of 'universal' use for the SMON problem. For fixed bids t_{-i}, of all other players, the allocations to a single player i depending on his own 2-dimensional bid vector t_i, partition the bid-space according to one of the shapes in Figure 1 (due to WMON). The positions of the boundary lines in these figures correspond to the (differences of the) payments to player i for the different allocations. We investigate these boundary positions (i.e. the truthful payments), as functions of the other players' bids.[5] For SMON mechanisms, Lemma 2 implies the linearity of these payment functions in 'most' cases. We prove that linearity of all boundary functions results in the mechanism being an affine minimizer. The only exceptions where linearity needs not hold, imply either a task independent mechanism or a boundary between different player-groups in a grouping minimizer (see Figures 2 and 3).

Our technical assumptions are *decisiveness, continuity* of the payment functions, and that the costs t_{ij} can take arbitrary real values.[6] For a discussion on these, and some examples of degenerate allocation rules, see the Appendix.

Preliminaries. The following notation and observations apply to any number of tasks. Since we treat the two-tasks case in the paper, we will illustrate these notions for $m = 2$. For a more detailed treatment see, e.g., [Christodoulou *et al.*, 2008; Vidali, 2009].

An *allocation matrix* is $a = (a_1, a_2, \ldots, a_n)$, where a_k is the binary *allocation vector* of player k. We also use α, $\alpha' \ldots$ etc. to denote some arbitrary m-dimensional allocation vector (for two tasks, $a_k \in \{(00), (01), (10), (11)\}$). For two tasks, a^{ik} denotes the allocation giving the first task to player i and the second to player k (mind the difference to a_{ij}, which is a single bit).

The bid matrix of all players except for player k, is denoted by t_{-k}, whereas t_{-ik} denotes the bid matrix of all players except for players i and k. For fixed t_{-k}, the *allocation regions* $R^k_\alpha = \{t_k \,|\, a_k(t_k, t_{-k}) = \alpha\} \subset \mathbb{R}^m$ for all possible α, partition the bid space $\mathbb{R}^m$ of player k into at most 2^m parts.

Definition 3. *An allocation function A satisfies SMON if $A(t_k, t_{-k}) = a$, and $A(t'_k, t_{-k}) = a'$, where $a \neq a'$, imply $(a_k - a'_k)(t_k - t'_k) < 0$ (≤ 0).*

For WMON allocations the allocation regions of any player must have a special geometric shape: the *boundary* between any two regions R^k_α and $R^k_{\alpha'}$ (if it exists) is on a hyperplane

$$(\alpha - \alpha') \cdot t_k = f^k_{\alpha:\alpha'},$$

where the functions $f^k_{\alpha:\alpha'}$, that determine these boundary positions are defined as follows (see [Vidali, 2009]): In a truthful mechanism, the payment of player k depends on t_{-k} and on a_k. Let $p^k_{a_k}(t_{-k})$ denote this payment. Then $f^k_{\alpha:\alpha'} = p^k_\alpha(t_{-k}) - p^k_{\alpha'}(t_{-k})$. In general for some t_{-k} some of the allocation areas R_α might disappear. We make the assumption that the allocation figures are complete (all the 2^m regions are always nonempty), i.e., the allocation is *decisive*. Thus, for WMON allocations of two tasks, the allocation of player k as a function of (t_{k1}, t_{k2}) has a geometrical representation of one of three possible shapes (see Figure 1). For given fixed t_{-k}, we call this geometric representation the *(allocation) figure of player k*.

In Section 2.1 we investigate how the positions $f^k_{\alpha:\alpha'}$ of a player's boundaries change as a function of t_{-k}. We will show that this change is linear in t_{-k} with only a few exceptions. For fixed t_{-ik} the boundary $f^k_{\alpha:\alpha'}$ is a function of the bid t_i. We assume the continuity of $f^k_{\alpha:\alpha'}(t_i)$ for any fixed t_{-ik}. Most of the time, w.l.o.g. we consider the figure and boundaries of the first player. In this case, for $k = 1$, we omit the superscript in f^k, R^k, etc.

In the rest of the section we summarize further implications of the SMON property, the strict version of WMON. See the Appendix for omitted proofs. If an allocation rule $A(t)$ is SMON, then the allocation of *all* players is constant in the interior of any region R^k_α (otherwise, changing t_k would result in changing a to a', with $a_k = a'_k$, contradicting SMON. We denote by $f^k_{a:a'}(t_{-k})$ the boundary between allocations a, a' (these describe the allocation of *all* players, not just of player k) for every t_{-k} *for which such a boundary exists* in the figure of player k. Next we state a crucial elementary property of continuous SMON mechanisms:

($\star$) *For fixed bids of the other players, the boundary $f^k_{a:a'}$ in the allocation figure of player k, considered as a function of the bids t_i of any particular player $i \neq k$ depends only on $(a'_i - a_i) \cdot t_i$, by some strictly increasing continuous function. If $a'_i = a_i$ then the boundary position is independent of t_i.*

For example, let $a = (11, 00, 00)$, and $a' = (01, 10, 00)$, and t_3 be fixed; then $f_{a:a'}(t_2) = \varphi(t_{21})$, for a nondecreasing real function φ. Similarly, if $a = (10, 01, 00)$ and

[4] I.e. every allocation occurs for at least one input t.

[5] Truthful payments are determined by the allocation function.

[6] Note that similar assumptions are made in [Dobzinski and Nisan, 2011; Dobzinski and Sundararajan, 2008; Christodoulou *et al.*, 2008; Mishra and Quadir, 2012].

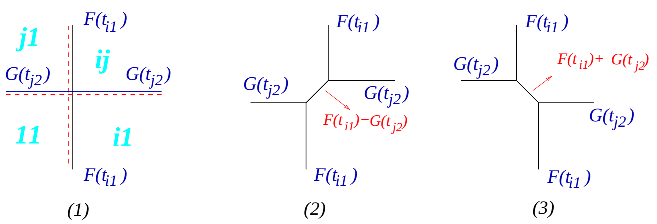

Figure 2: Types of quasi-independent allocation figures of player 1. Here $j = i$ is allowed. Only in type (1) can F or G be locally nonlinear.

$a' = (01, 10, 00)$, then $f_{a:a'} = \psi(t_{21} - t_{22})$ for some nondecreasing function ψ. The property ($\star$) is implied by Lemma 1, and Observation 1 below:

Lemma 1. *[increasing boundaries] Let t_{-1i} be fixed, and $\mathcal{G} \subset \mathbb{R}^m$ a connected set. Assume that the boundary $f_{a:a'}$ exists for all $t_i \in \mathcal{G}$. For every $t_i, \bar{t}_i \in \mathcal{G}$ it holds that if $(a'_i - a_i) \cdot t_i < (a'_i - a_i) \cdot \bar{t}_i$, then $f_{a:a'}(t_i) \leq f_{a:a'}(\bar{t}_i)$.*

Corollary 1. *Let S be a subset of the tasks. If $a_{ij} = a'_{ij}$ for all $j \in S$, then $f_{a:a'}(t_i) = f_{a:a'}(\bar{t}_i)$ whenever t_i and $\bar{t}_i$ differ only on tasks in S.*

For example, let $a = (11, 00, 00)$, and $a' = (01, 10, 00)$. By the lemma (for $i = 2$), $f_{11:01}$ is increasing in t_{21}. The corollary says, that $f_{11:01}$ is a function of *only* t_{21}. S consists of task $j = 2$, because the allocation of this task to player 2 remains the same $a_{22} = a'_{22} = 0$. Consider t_2 and $\bar{t}_2$ bids such that $t_{21} = \bar{t}_{21}$. If $f_{11:01}(t_2) < f_{11:01}(t'_2)$ were the case, then by the lemma, $f_{11:01}$ would have a jump of at least $f_{11:01}(t'_2) - f_{11:01}(t'_2)$ in t_{21}. However, $f_{11:01}(t_{21})$ is continuous by assumption. We conclude the subsection with a couple of simple observations that hold in the case of continuous boundary functions.

Observation 1. *(a) [boundary points] Let $t_{-1} = (t_i, t_{-1i})$. If t_1 is a point on the boundary $f_{a:a'}(t_{-1})$, where the allocations a_i and a'_i are different, then for $t_{-i} = (t_1, t_{-1i})$ the bid t_i is a point on the boundary $f^i_{a:a'}(t_{-i})$.*

(b) [inverse boundaries] Let t_{-1i} be fixed. If for some monotone continuous univariate function $f_{a:a'}(t_i) = \varphi((a'_i - a_i) \cdot t_i)$, then $f^i_{a':a}(t_1) = \varphi^{-1}((a_1 - a'_1) \cdot t_1)$.

(c) [strictly increasing boundaries] Let t_{-1i} be fixed, and $\mathcal{G} \subset \mathbb{R}^m$ a connected set. Assume that the boundary $f_{a:a'}$ exists for all $t_i \in \mathcal{G}$. The function $f_{a:a'}(t_i)$ is strictly increasing in $(a'_i - a_i) \cdot t_i$ over $t_i \in \mathcal{G}$.

2 Characterization

In this section we sketch the proof of Theorem 1. The Appendix contains the detailed proofs. Figures 2 and 3 show the possible allocation figures of player 1, w.r.t. the dependence of boundaries on other players' bids. The player indices marking the four regions indicate the players who get the tasks in the respective region.[7] In every allocation figure, task 1 is either given to the same player in R_{01} and in R_{00}, or to two different players (and similarly for task 2). Accordingly, the boundaries where task 1 changes owner are either only the vertical boundaries (like in types (1) to (6)), or the vertical boundaries and $f_{01:00}$ where two other players exchange the task among each other. Combining all possibilities for both tasks, we obtain the depicted cases.

Definition 4. *We call the allocations in an allocation figure* quasi-independent, *if the same player receives task 1 in R_{01} and in R_{00}, and the same (possibly other) player receives task 2 in R_{10} and in R_{00}.*

Figures 2 and 3 show all possible *quasi-independent* and *non quasi-independent* allocations, respectively, up to symmetry. Observe that for a particular boundary ($\star$) only implies that it(s position) is a multivariate function, monotone in each variable. (E.g., in case (6) in Figure 3 $f_{10:00}$ is some function $f(t_{i1}, t_{j2}, t_{k2})$.) However, whenever different *other* boundaries depend on these different variables, it must be the case that the multivariate function is a so called *additively separable* function as appears in the figure (e.g., in (6), if only t_{i1} is increased (locally), then only the two vertical boundaries move, and $f()$ as a function of t_{i1} is necessarily of the form $F(t_{i1}) + C(t_{j2}, t_{k2})$ for some function F; by similar considerations, G and H functions exist s.t. $f = F(t_{i1}) - G(t_{j2}) + H(t_{k2}) + C$).

2.1 Local linearity results

The main theorems of this subsection show the linearity of different boundary functions $f_{\alpha:\alpha'}(t_i)$. Since task-independent allocations, that are in general nonlinear, have "crossing" figures (see Figure 1 (b)), it will be important to distinguish two types of allocation figures:

Definition 5. *For fixed t_{-k}, the allocation figure of k is* crossing, *if $f^k_{11:01} = f^k_{10:00}$, and $f^k_{11:10} = f^k_{01:00}$. Otherwise we call the figure* non-crossing.

Mechanisms with only crossing allocation figures are task-independent. The boundary functions $f_{\alpha:\alpha'}$ of such mechanisms need not be linear, as they indicate the critical values for getting the task in arbitrary monotone single-task allocations. In what follows, we show that if the mechanism is onto, the converse also holds: if the mechanism has an allocation figure (for some k and t_{-k}) that is non-crossing, then *all* boundary functions of the mechanism must be linear. However, if the mechanism is not onto, that is, certain allocations a^{ij} never occur, then even complete (decisive) non-crossing allocation figures might change by non-linear functions. The resulting mechanisms, which we named (player) grouping minimizers, constitute a generalization of affine minimizers. The main reason for enforced linearity of the boundaries is given in the following basic lemma. Namely, whenever these boundaries are additive separable functions of at least two variables, and so are their inverse boundaries (on another

[7] As an example, consider Figure 2 (1). Here player i gets task 1 in regions R_{01} and R_{00}; and player j gets task 2 in R_{00} and R_{10}. By property ($\star$), the position $f_{11:01}$ of the boundary between R_{11} and R_{01} is given by an increasing function of t_{i1}, that we denote by $F(t_{i1})$. Similarly, the horizontal boundary position is determined by an increasing function $G(t_{j2})$. If the figure has a shape like in Figure 2 (2), then the slanted boundary has the equation $t_{11} - t_{12} = F(t_{i1}) - G(t_{j2})$.

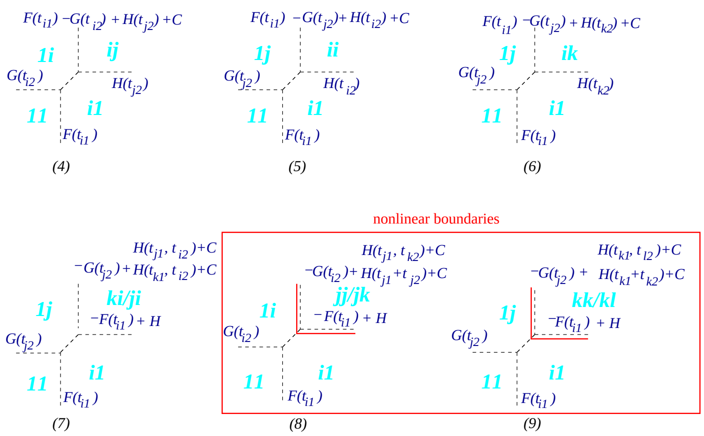

Figure 3: *Non*-quasi-independent allocation figures of player 1. Here the shapes of the figures are arbitrary. The letters indicate the players who receive the two tasks in the respective regions; $i, j, k, \ell \neq 1$ denote different players. F and G are linear, and only in (8) and (9) can H be locally nonlinear.

player's figure), we encounter a situation that fulfil the conditions of the lemma. The conditions imply that if the functions β and ψ are monotone, then for *any* small enough Δ, the second curve is a parallel translation of the first one *both* in vertical and in horizontal direction. This is possible only if the curve is a straight line (α and φ are linear).

Lemma 2. *Assume that for strictly monotone continuous real functions α, β, φ, and ψ, for every $(x, y, z) \in \mathcal{G}$ for an open set $\mathcal{G} \subseteq \mathbb{R}^3$, it holds that*
$(y = \alpha(x) + \beta(z)) \Leftrightarrow (x = \varphi(y) + \psi(z).)$ *Moreover, we assume that an open neighborhood of (x, z) pairs exists for which $(x, \alpha(x) + \beta(z), z) \in \mathcal{G}$. Then*

(a) α and φ are linear functions;

(b) α and φ are both increasing or both decreasing, and exactly one of β and ψ is increasing;

(c) if β and ψ are also linear functions with slopes λ_β and λ_ψ, then the slopes λ_a of α and λ_φ of φ satisfy $\lambda_\alpha = -\frac{\lambda_\beta}{\lambda_\psi}$, and $\lambda_\varphi = -\frac{\lambda_\psi}{\lambda_\beta}$.

Definition 6. *The real function $\varphi : \mathbb{R} \to \mathbb{R}$ is* locally linear in point x, *if a $\delta > 0$ exists such that φ is linear in the interval $(x - \delta, x + \delta)$; φ is* locally non-linear in x, *if φ is not a linear function in any open neighborhood of x.*

With the help of Lemma 2, in Theorems 2 and 3 we show the local linearity of the boundary functions of region R_{11} (F and G) and also of R_{00} (H) in most cases. The exceptional cases, when linearity in general does not hold, are case (1) for functions F and G, which is the allocation of a task-independent mechanism, and cases (8) and (9) for H which are typical allocations of grouping minimizers. Notice that (8), (9) are exactly the allocation types, where in region R_{00} the tasks are given only to players who do not get a job in any other region.

Theorem 2. *If an allocation figure of player 1 is constant (i.e. the allocations of* all *players in each region are constant) for some connected open set $\mathcal{G} \subset \mathbb{R}^{(n-1)\times 2}$ of t_{-1} values, then the function F (resp. G) is locally linear in every point t_{i1} (resp. t_{i2} or t_{j2}) in the projection of $\mathcal{G}$, or case (1) of Figure 2 holds over $\mathcal{G}$.*

Theorem 3. *If an allocation figure of player 1 is constant for some connected open set $\mathcal{G} \subset \mathbb{R}^{(n-1)\times 2}$ of t_{-1} values, then the function $H()$ (when defined) is locally linear in each variable of H, in every point in the corresponding projection of $\mathcal{G}$, or else one of the cases (8) or (9) in Figure 3 holds.*

2.2 Global linearity results

Next, we focus on the functions F and G. We prove that local linearity extends to global linearity of these functions. More precisely, the functions $F(t_{i1})$ for all $i \neq 1$ are uniquely determined with domain $\mathbb{R}$ (unless the allocation a^{i1} does not occur at all), and the same holds for $G(t_{i2})$. Moreover, they prove to be linear, unless the mechanism is task-independent. It turns out that the requirement of Theorem 2 to have constant allocations in *all the four* regions is not necessary, concerning the functions F and G. The next observation extends Corollary 1 to the case when other players' bids are not fixed.

Observation 2. *Let $t_{-1} = (t_2, t_3, \ldots, t_n)$ and $t'_{-1} = (t'_2, t'_3, \ldots, t'_n)$, so that $t_{i1} = t'_{i1}$. If both boundaries $f_{a^{11}:a^{i1}}(t_{-1})$ and $f_{a^{11}:a^{i1}}(t'_{-1})$ exist, then they are equal.*

From now on, we can use $F^1_i(t_{i1})$ and $G^1_i(t_{i2})$ (in general $F^k_i(t_{i1})$ and $G^k_i(t_{i2})$) to denote the over their whole domain uniquely determined boundary functions of a mechanism. We will omit the superscript 1 whenever we consider the allocation of player 1. We omit the subscript i, when it is clear from the argument t_{i1} or t_{i2}. We prove that if the mechanism is not task-independent, these functions are linear and have domain $\mathbb{R}$ or $\emptyset$. The proof uses the following lemmas:

Lemma 3. *If all allocation figures of a single player are crossing, then all allocation figures of all players are crossing, and therefore the mechanism is task-independent.*

Lemma 4. *Let $\bar{t}_{i1}$ be an interior point of the domain of $F(t_{i1})$. If F is locally non-linear in $\bar{t}_{i1}$, then the mechanism is task-independent.*

Theorem 4. *If the SMON mechanism is not task-independent, then in the allocation figures of player 1,*

(a) every F_i and every G_i function is linear (when defined);

(b) for any fixed t_{-1} the boundaries of the region R_{11} are $f_{11:01} = \min_{i\neq 1} F_i(t_{i1})$ and $f_{11:10} = \min_{i\neq 1} G_i(t_{i2})$;

(c) the domain of every F_i and every G_i is $\mathbb{R}$ or $\emptyset$;

(d) if a^{i1} exists then for any fixed t_{-1}, the t_{j1} values of the players $j \neq 1, i$ can be increased so that in R_{01} the allocation becomes a^{i1}; in turn, for any fixed t_{-1}, the t_{i1} can be decreased so that in R_{01} the allocation becomes a^{i1} (and similarly for t_{i2}).

The same holds for the allocation figures of every player k.

2.3 Non task-independent mechanisms

This subsection completes the characterization. We define the slopes of different boundary functions and settle the connection between them. For ease of exposition, henceforth we assume that the SMON mechanism we consider is *not* task-independent (more precisely, not threshold), and therefore has at least one non-crossing allocation figure by Lemma 4. By Theorem 4, the $F_i^k(t_{i1})$ and $G_i^k(t_{i2})$ functions are linear over the whole real domain. We introduce the notation:

Notation 1. *If the allocation a^{ik} ever occurs in the mechanism, then we denote the slope of the linear function $F_i^k(t_{i1})$ by $\lambda_{i,horiz}^k$; if the allocation a^{ki} occurs, we denote the slope of the function $G_i^k(t_{i2})$ by $\lambda_{i,vert}^k$.*

Later we will prove that for non task-independent allocations $\lambda_{i,horiz}^k = \lambda_{i,vert}^k$ must hold. Before showing this, it will be useful to first elaborate on the H functions (see Figure 3). These will turn out to be linear, unless the players are partitioned into isolated groups that never share the jobs (in particular, H is always linear in cases (4)–(7), but not necessarily in cases (8) and (9)). Note also that such H functions never occur in task-independent allocations. We treat the allocation figures of player 1, and first examine the dependence of H functions on the bids of players i, with whom player 1 sometimes shares the jobs, i.e., either of a^{i1} or a^{1i} occurs as allocation. This case will also serve as the base case of the subsequent induction proof of Theorem 6.

Theorem 5. *Assume that the allocation a^{i1} occurs in the mechanism. Then whenever a boundary function $H()$ of the region R_{00} depends on t_{i1}, or t_{i2}, or on $t_{i1}+t_{i2}$, this dependence is linear with slope $\lambda_{i,horiz}^1$. Analogously, if the allocation a^{1i} exists, then for any of these arguments the function has slope $\lambda_{i,vert}^1$.*

Corollary 2. *If for some open set of t_{-k} values, the allocation to player k is of type (4)–(9) so that H depends on t_{i1} or t_{i2} or on $t_{i1}+t_{i2}$, then $\lambda_{i,horiz}^k = \lambda_{i,vert}^k$ (given that both are defined).*

Lemma 5. *For any SMON allocation $\lambda_{i,horiz}^k = \lambda_{i,vert}^k$ (when defined), unless the allocation is task-independent.*

Next we define a partition of the players, such that restricted to any set of the partition, the mechanism is an affine minimizer (given that the tasks are allocated to the respective set of players).

Definition 7. *We define the* player-graph *with the set of players $[n]$ as vertices: let players i and j be connected by an edge if the allocation a^{ij} occurs in the mechanism. The players of the same connected component are called a* group.

For *neighboring* players i and k in the player-graph by Lemma 5 we can define $\lambda_i^k = \lambda_{i,horiz}^k = \lambda_{i,vert}^k$. Observe that for these λ_i^k values $\lambda_i^k = 1/\lambda_k^i$ is obvious by Observation 1 (b).

Lemma 6. *Assume that the mechanism is not task-independent and i, j, k is a triangle in the player-graph. Then $\lambda_i^k = \lambda_j^k \cdot \lambda_i^j$.*

The next theorem shows that the constant slopes λ_i^k can be defined for *any* pair of players within the same group.

Theorem 6. *Assume that the mechanism is not task-independent. For any two players i and k of the same group there exist constants $\lambda_i^k = 1/\lambda_k^i$ such that in every allocation figure of player k where the functions F_i^k, G_i^k appear or H depends on t_i, they depend linearly on t_i with slope λ_i^k.*

Observation 3. *If the mechanism is not task-independent, then for an arbitrary path $i_0, i_1, i_2, \ldots, i_t$ in the player-graph it holds that $\lambda_{i_t}^{i_0} = \lambda_{i_1}^{i_0} \cdot \lambda_{i_2}^{i_1} \cdot \ldots \cdot \lambda_{i_t}^{i_{t-1}}$.*

Corollary 3. *For any three players i, j, k of the same group, $\lambda_i^k = \lambda_j^k \cdot \lambda_i^j$.*

Due to the uniqueness and transitivity of the λ values in non task-independent allocations, we can choose $\lambda_i = \lambda_i^1$ to be the multiplicative weight of a player i in the group of player 1. (We can choose a representative player in each group to play the role of player 1.) In order to determine the affine minimizer within a connected group of players completely, we need additive constants for each allocation to these players, which we define next.

Lemma 7. *Consider w.l.o.g. the connected group of player 1. There exist constants c^{ii} and c^{ij} for arbitrary members i and j of the group, such that within this group the mechanism allocates according to an affine minimizer with multiplicative constants λ_i^1 and additive constants c^{ii} and c^{ij} (given that this group of players receives the tasks).*

Clearly, for every connected group g we can choose a representative player k_g, and determine the multiplicative and additive constants $\lambda_i^{k_g}$, c^{ij} and c^{ii} accordingly. We assume that $k_1 = 1$. It remains to elaborate on the rules of the allocation between two different groups of players. Let Opt_g be the optimum value of group g, that is $Opt_g = \min_{i,j\in g}(\lambda_i^{k_g} t_{i1} + \lambda_j^{k_g} t_{j2} + c^{ij})$. The characterization result (Theorem 1) is now an immediate corollary of Lemma 8.

Lemma 8. *For every connected group g of players, there exists an increasing continuous function Φ_g with domain $(-\infty, C_g)$ and $\lim_{-\infty} \Phi_g = -\infty$ s.t. (optimal players of) group g with minimum value of $\Phi_g(Opt_g)$ receive the tasks.*

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A Notes on our assumptions and discussion of the result

Our technical assumptions are *decisiveness, continuity* of the payment functions, and that the costs t_{ij} can take arbitrary real values. These assumptions facilitate a fairly compact characterization result. The following examples illustrate that such simplicity of the result is unthinkable by admitting either discontinuities or non-decisive mechanisms, whereas most probably there would be no gain by such an extension (e.g., concerning lower bounds).

1. We require decisiveness. Geometrically this means that all allocation figures are always complete; in particular, for $m = 2$ they always have the four allocations regions 00, 10, 01, and 11.

Definition 8. *An allocation function A is* decisive*, if every player i, for every fixed bids t_{-i} of the other players and every particular allocation $\alpha \in \{0,1\}^m$ has bids t_i so that A allocates him exactly the items in α.*

Decisiveness is a natural assumption [Dobzinski and Sundararajan, 2008; Christodoulou *et al.*, 2008]. Without it, it is easy to define degenerate allocations with arbitrary (increasing) payment functions. An example is an allocation function where the tasks are bundled, and are always allocated to a single player.

2. Another assumption we make is that the payments (i.e., boundary positions) of the mechanism are continuous functions. This is not as severe an assumption since they are anyway non-decreasing functions. The following example shows that without the continuity assumption, unnatural mechanisms exist even for *two tasks and two players*, that are not task-independent, threshold or affine minimizers. (In this example the allocation figure of player 1 is non-crossing only for a single bid t_2 of player 2):

Example 2. *Let $n = m = 2$, and consider the following (non-continuous) task-independent mechanism with $F(t_{21})$ and $G(t_{22})$ as vertical and horizontal boundary functions in the allocation figure of player 1: $F(t_{21}) = t_{21}$ if $t_{21} \leq 1$ and $F(t_{21}) = t_{21} + 1$ if $t_{21} > 1$. Note that the (vertical) boundary function has a jump of 1 in $t_{21} = 1$. Analogous rule applies to task 2 and the horizontal boundary.*

We will refine the allocation rules by modifying them in $t_{21} = 1$ and/or $t_{22} = 1$. Let $F(1, t_{22}) = 1$ if $t_{22} < 1$, and $F(1, t_{22}) = 2$ if $t_{22} > 1$, (and symmetrically for $G(t_{21}, 1)$. Finally, if $t_{21} = t_{22} = 1$, then the allocation figure of player 1 is non-crossing with $f_{11:01} = f_{11:10} = 1$, and $f_{10:00} = f_{01:00} = 2$. Note that every other allocation figure of both players is crossing.

3. A less natural assumption is that the t_{ij} can take positive *and* negative values. This assumption was made in [Christodoulou *et al.*, 2008], and in [Dobzinski and Sundararajan, 2008] (for auctions). Both of these previous results and this paper suggest the strong conjecture that (even for more players) task-independent mechanisms are the only decisive mechanisms when we admit only positive (scheduling) or only negative (auctions) bids.[8] Therefore it seems that

[8] The intuitive explanation is, that if the domain is bounded, then

we would either have to give up decisiveness or possibly examine only task-independent allocations. We think that we can learn more about the behaviour of monotone allocations in general (or, for 'high enough' bids in the scheduling or auctions domain) by investigating real bids first. In particular, note that (as opposed to the other two assumptions) we do not only *restrict*, but also *extend* the class of investigated allocations.

The following example shows that if the t_{ij} bids are not allowed to take arbitrary real values, then continuous monotone (albeit not strongly monotone) allocations exist that are not grouping minimizers. The example is for the combinatorial auctions domain ($t \in \mathbb{R}_-$), for $m = 2$, and $n \geq 3$. A very similar example was given in [Lavi *et al.*, 2003] (see Example 4. there). However, no analogous allocation has been found for the scheduling domain (moreover, these examples are not strongly monotone), so that deeper investigation would be interesting. The positive and negative orthants (scheduling vs. auctions) behave differently, since in both cases the smaller – or 'larger' negative – values receive the tasks; that is, we mirror the possible valuations, but not the allocation rules. This explains why results can be obtained parallel in both settings, and also, why these can be different.

Example 3. *Assume that $t_{i1} + t_{i2} \leq t_{j1} + t_{j2} \leq t_{k1} + t_{k2} \leq \ldots$ are the three smallest sums of bids over all players (break ties by player indices). Let $\alpha = -(t_{k1} + t_{k2}) \geq 0$. The two jobs are allocated to players i and j, according to an affine minimizer with $c^{ji} = c^{ji} = \alpha$, and $c^{ii} = c^{jj} = 0$. This mechanism is well-defined, and truthful. Note that for player i the area $t_{k1} + t_{k2} < t_{i1} + t_{i2}$ must be part of the region R^i_{00}. (It is crucial that this area is bounded in the negative orthant, but not bounded if t_{ij} can take positive values.) This mechanism is onto, but not decisive: as t_j approaches t_k, the regions R^i_{10} and R^i_{01} disappear, and j and k can change roles without any problem.*

A.1 Discussion of the result

The most immediate open question is, whether our result extends to SMON mechanisms with many tasks. The characterization can be generalized if we make the following strong assumption: for any two tasks u and v, if two players (ever) share these tasks in the 2-dimensional projection allocation obtained by fixing the bids for every other task to some t^{-uv} matrix, then they also share the tasks given any other fixed values t'^{-uv}. This property holds for all the mechanisms that we know, but it is not clear why it can be assumed right away. We formulate the following strong conjecture:

Conjecture 1. *Every continuous decisive SMON mechanism for allocating m items, with additive bidder valuations and $t_{ij} \in \mathbb{R}$, is the product of grouping minimizers and a task-independent mechanism. That is, the set of tasks M can be partitioned $M = \{M_0, M_1, \ldots, M_s\}$ such that on the tasks of M_0 the mechanism is task-independent (threshold), and on every other partition M_q it is an arbitrary grouping minimizer.*

non-crossing allocation figures become incomplete as soon as the slanted boundary of the figure "reaches" the border (where $t_{ij} = 0$), and therefore these mechanisms are inherently non-decisive.

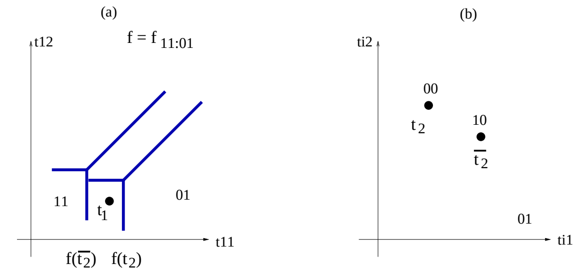

Figure 4: Illustration to Lemma 1. (a) Allocation figure of player 1 for t_i and for $\bar{t}_i$; (b) the figure of player i for t_1

Another, even more challenging question is, if Lemma 2 can in any way be helpful to gain insight into the nature of WMON allocations. This could be the case, since the lemma is based on local properties of the boundary functions, and similar properties must hold in the WMON case as well (that the boundary may depend on further other player's bids should actually impose just further restrictions). Moreover, the lemma suggests the intuition, that in order to obtain new types of WMON allocation rules, we need to find ones where the boundary functions are *not* additive separable functions of the relevant variables.[9]

B Omitted proofs

Proof of Lemma 1. An intuitive proof of the lemma is the following (see figure 4). Let $m = 2$, $a = a^{11}$ and $a' = a^{i1}$, and consider the boundary $f_{11:01}$ in the figure of player 1. Note that $a_i = (00)$ and $a'_i = (10)$, and $(a'_i - a_i) \cdot t_i < (a'_i - a_i) \cdot \bar{t}_i$, means $t_{i1} < \bar{t}_{i1}$ in this case. Assume for contradiction that $f_{11:01}(t_i) > f_{11:01}(\bar{t}_i)$. Then there is a $t_1 \in \mathbb{R}^2$ such that $f_{11:01}(\bar{t}_i) < t_{11} < f_{11:01}(t_i)$, and so that the allocation for (t_1, t_i, t_{-1i}) is a^{11} but for $(t_1, \bar{t}_i, t_{-1i})$ it is a^{i1}. Thus, $t_i \in R^i_{00}$ whereas $\bar{t}_i \in R^i_{10}$ for this t_1. A contradiction, since for monotone allocations no point in R^i_{00} can have smaller t_{i1} than any point in R^i_{10}.

Proof of Observation 1 (a) is a straightforward consequence of continuity. (b) is now immediate, since 'being a point on the boundary' means $f_{a:a'}(t_i) = (a_1 - a'_1) \cdot t_1$. (c) If a continuous increasing function is not strictly increasing, then it is constant over an interval. However, then the inverse function (which is another boundary function by (b)) would have a jump, contradicting continuity.

Proof of Lemma 2. We consider an open rectangular set $(x_0, x_1) \times (y_0, y_1) \times (z_0, z_1) \subset \mathcal{G}$, such that if $x \in (x_0, x_1)$ and $z \in (z_0, z_1)$ then $\alpha(x) + \beta(z) \in (y_0, y_1)$. It can be verified that such a set exists by the condition of the lemma. Since the functions $\alpha, \beta, \varphi, \psi$ are strictly monotone and continuous, they are invertible, so the condition of the lemma can be rewritten as $y = \varphi^{-1}(x - \psi(z))$ and combining the latter with (1) we get

$$\alpha(x) + \beta(z) = \varphi^{-1}(x - \psi(z)). \qquad (2)$$

Note that here we exploit that $y = \alpha(x) + \beta(z) \in (y_0, y_1)$. From the last equation, by fixing z we can see that α and φ

[9]E.g., are not of the form $F(t_{i1}) + G(t_{j2})$, like in Figure 2 (3), but of some other $H(t_{i1}, t_{j2})$.

are either both increasing or both decreasing, which proves (b). Assume w.l.o.g. that α and φ are increasing.

Fix an arbitrary $z \in (z_0, z_1)$. Suppose first that ψ is decreasing. By the continuity and monotonicity of ψ, for every small enough $\Delta > 0$ there exists a $\delta > 0$ such that $\psi(z) - \psi(z+\delta) = \Delta$. Moreover, we can choose $\Delta > 0$ small enough so that also $z + \delta < z_1$ holds. Now for an *arbitrary* $x \in (x_0, x_1 - \Delta)$ we have

$$\begin{aligned}\alpha(x+\Delta) + \beta(z) &= \varphi^{-1}(x + \Delta - \psi(z)) \\ &= \varphi^{-1}(x - \psi(z+\delta)) \\ &= \alpha(x) + \beta(z+\delta);\end{aligned}$$

where the first and third equality follows from (2), and the second by the definition of δ. Rearranging the first and the last terms, we obtain

$$\alpha(x+\Delta) - \alpha(x) = \beta(z+\delta) - \beta(z).$$

Note that the right hand side is constant, but $x \in (x_0, x_1 - \Delta)$ was arbitrary. Moreover, since (a small) Δ can be chosen arbitrarily, we showed that the increase of $\alpha()$ is constant for constant increase of x, independently of x itself. It is easy to see that such an $\alpha()$ must be linear (e.g., by halving intervals recursively, the function values of the two endpoints must be averaged every time. The linearity of φ follows analogously. If ψ is increasing, then the same proof holds with $\delta < 0$. Observe that if ψ is decreasing, then δ is positive, so β is increasing (since α is increasing), and vice versa. Finally, assume that β and ψ are linear. Then $\lambda_\beta \cdot \delta = \lambda_\alpha \cdot \Delta$. Moreover, δ was chosen so that $\Delta/\delta = -\lambda_\psi$. This yields $\lambda_\alpha = -\frac{\lambda_\beta}{\lambda_\psi}$, and the other statement of (c) follows analogously.

Proof of Theorem 2. We assume an open rectangle $\mathcal{T} \subset \mathcal{G}$ where the conditions hold. Then the result extends to an arbitrary open set $\mathcal{G}$. Here we give a detailed proof of the theorem in cases (2) and (9) of Figure 3 as examples for how to apply Lemma 2 for the linearity of boundary functions. The other cases can be shown analogously.

In case (2), the line of the (slanted) boundary $f_{10:01}$ is given by the equation $t_{11} - t_{12} = F(t_{i1}) - G(t_{j2})$, and it is a boundary between allocations a^{1j} and a^{i1}. If we fix a boundary point t_1, then for this t_1 the figure of player i has a boundary $f^i_{10:00} = f^i_{a^{i1}:a^{1j}}$, and t_i is a point of this boundary by Observation 1 (b). Moreover, the boundary position for i is a function of the form $t_{i1} = -G^i(t_{12}) + H^i(t_{11}, t_{j2})$, for some monotone increasing functions G^i and H^i. For fixed $\bar{t}_{j2}$ we obtain

$$t_{12} = -F(t_{i1}) + t_{11} + G(\bar{t}_{j2})$$
$$\Updownarrow$$
$$t_{i1} = -G^i(t_{12}) + H^i(t_{11}, \bar{t}_{j2}),$$

and this is valid in some neighborhood of (t_{11}, t_{12}, t_{i1}), since it is valid on $t_{-1} \in \mathcal{T}$.

We set $y := t_{12}$, $x := t_{i1}$, $z := t_{11}$ and choose the strictly monotone, continuous functions $-F(x)$, $z + G(\bar{t}_{j2})$, $-G^i(y)$, and $H^i(z, \bar{t}_{j2})$, as $\alpha()$, $\beta()$, $\varphi()$, and $\psi(z)$, respectively. Applying the lemma yields that F is locally linear. For proving the linearity of G we must fix t_{i1} instead of t_{j2}.

Next, we consider case (9), and the boundary $f_{01:00} = f_{a^{i1}:a^{k\ell}}$. Here we have for some fixed $\bar{t}_{\ell 2}$

$$t_{12} = -F(t_{i1}) + H(t_{k1}, \bar{t}_{\ell 2})$$
$$\Updownarrow$$
$$t_{i1} = -G^i(t_{12}) + H^i(t_{k1}, \bar{t}_{\ell 2}),$$

and by Lemma 2 the linearity of F follows. In order to prove the linearity of G, one needs to consider the boundary $f_{10:00}$.

Proof of Theorem 3. We prove the theorem for all of cases (4) – (7). The proofs are again direct applications of Lemma 2; the proof technique is the same as for Theorem 2.

In case (4) we consider the boundary $f_{10:00}$ between allocations a^{1i}, and a^{ij}. For t_1 points on the boundary we have $t_{11} = F(t_{i1}) - G(t_{i2}) + H(t_{j2}) + C$. In the figure of player j this corresponds to the boundary $f^j_{01:00}$, and there $t_{j2} = -F^j(t_{i1}) + H^j(t_{11}, t_{i2})$. We fix $\bar{t}_{i2}$, and obtain

$$t_{11} = H(t_{j2}) + F(t_{i1}) + C - G(\bar{t}_{i2})$$
$$\Updownarrow$$
$$t_{j2} = H^j(t_{11}, \bar{t}_{i2}) - F^j(t_{i1});$$

so $H(t_{j2})$ is linear.

In case (5), we look at the same boundary $f_{10:00}$, and with fixed $\bar{t}_{j2}$ use the formula for boundary points $t_{11} = H(t_{i2}) + F(t_{i1}) + C - G(\bar{t}_{j2})$. In the figure of player i is this corresponds to a boundary point on the slanted boundary $f^i_{11:00}$. For this boundary it holds that $t_{i1} + t_{i2} = H^i(t_{11}, t_{j2})$. Therefore, we can use the equivalence

$$t_{11} = H(t_{i2}) + F(t_{i1}) + C - G(\bar{t}_{j2})$$
$$\Updownarrow$$
$$t_{i2} = H^i(t_{11}, \bar{t}_{j2}) - t_{i1},$$

and obtain the linearity of $H(t_{i2})$.

In case (6), for the same boundary $f_{10:00}$, we obtain linearity from

$$t_{11} = H(t_{k2}) + F(t_{i1}) + C - G(\bar{t}_{j2})$$
$$\Updownarrow$$
$$t_{k2} = H^k(t_{11}, \bar{t}_{j2}) - F^k(t_{i1}).$$

Finally, in case (7), consider again boundary $f_{10:00} = f_{a^{1j}:a^{ji}}$. When we fix $\bar{t}_{i2}$

$$t_{11} = H(t_{j1}, \bar{t}_{i2}) - G(t_{j2}) + C$$
$$\Updownarrow$$
$$t_{j1} = F^j(t_{11}) + t_{j2} - G^j(\bar{t}_{i2})$$

implies the linearity of H in the variable t_{j1}. In order to show linearity in t_{i2}, we use the symmetric boundary and $t_{12} = H(\bar{t}_{j1}, t_{i2}) - F(t_{i1})$. Assuming that in (7) H is of the form $H(t_{k1}, t_{i2})$, we use the equivalence

$$t_{12} = H(t_{k1}, t_{i2}) - F(t_{i1})$$
$$\Updownarrow$$
$$t_{i1} - t_{i2} = F^i(t_{k1}) - G^i(t_{12}),$$

that is apparent from the allocation figure of i. The linearity of H in t_{i2} is immediate. For the linearity in t_{k1} we need to exploit the linearity of $F^i(t_{k1})$, as follows from Theorem 2, and the fact that the boundary between a^{i1} and a^{ki} exists, so the figure of i is not of shape (1).

Proof of Observation 2. For every $k \neq 1$ the bid t_{k2} can be changed to $\max(t_{k2}, t'_{k2})$ without changing the allocation a^{11} of t_1 points in R_{11} and the allocation a^{i1} of points in R_{01}. Similarly, for $k \neq i, 1$ the bid t_{k1} can be changed to $\max(t_{k1}, t'_{k1})$. None of these changes modifies the allocations a^{11} and a^{i1} to any other allocation by SMON, so the position of the boundary between them remains $f_{a^{11}:a^{i1}}[t_{-1}]$. Since the same argument holds for changing the bids in t_{-1}, it must be the case that the boundary positions were equal $f_{a^{11}:a^{i1}}[t_{-1}] = f_{a^{11}:a^{i1}}[t'_{-1}]$.

Observation 4. *The domain of $F(t_{i1})$ is $(-\infty, c)$ or $(-\infty, c]$ for some $c \in \mathbb{R} \cup \{-\infty, +\infty\}$, and similarly for $G(t_{i2})$.*

Proof of Observation 4. If the boundary $f_{a^{11}:a^{i1}}$ exists for $t_{-1} = (t_2, \ldots, t_i, \ldots, t_n)$, then it exists for $t_{-1} = (t_2, \ldots, t'_i, \ldots, t_n)$, where $t'_{i1} < t_{i1}$, and $t'_{i2} = t_{i2}$ (in fact, by the previous observation, t'_{i2} and t'_k $(k \neq i, 1)$ can be arbitrary).

The following Observation implies that in general the allocations having to be constant for an open set of t_{-1} values is not too restrictive for continuous allocations. Similar observations can be made also for (G and) even for the H functions.

Observation 5. *If t_{i1} is in the interior of the domain of $F(t_{i1})$, then there exist t_{i2}, and t_k $(k \neq 1, i)$ bids, so that $t_{-1} = (t_2, \ldots, t_n)$ has an open neighborhood, where the four allocations in the figure of player 1 are constant.*

Proof of Observation 5. Since we have a finite number of variables, they can be moved (one by one) to some non-boundary point of their own allocation figures, so that the later moving of bids does not move a boundary of a previously fixed bid to be incident to the bid. Moreover, the allocation a^{i1} remains valid in the region R_{01} of player 1.

Proof of Lemma 3. Assume w.l.o.g. that it is player 1 who only has crossing allocation figures. As a first step, we show that all allocation figures of player 1 are quasi-independent (see Definition 4), disregarding tie-breaks. If the allocation figure is not quasi-independent, then it is one of types (4)–(9), i.e. the figure depends by some function H on a bid, that does not receive tasks in R_{01} and in R_{10}. If all allocations in the regions remain constant after we increase or decrease this bid, then only the boundaries of R_{00} change, and the figure becomes non-crossing, a contradiction. Therefore, if we increase this bid by an arbitrary small δ, it must lose the item in region R_{00}. Similarly, if we decrease this bid, it must receive the item in region R_{01} (resp. region R_{10}) if it was a bid for task 1 (resp. task 2). That is, the non quasi-independent allocation figure was due to different tie-breaking depending on t_{12} (resp. on t_{11}), and it is not valid in any open neighborhood of t_{-1}. Importantly, if a single entry, say t_{k1} is increased/decreased in an arbitrary t_{-1}, then the *only* possible change (if any) in the figure of player 1 is the increase/decrease of the vertical boundaries, and analogously a change in t_{k2} can only change the horizontal boundaries.

We suppose for contradiction that a player i exists with a non-crossing allocation figure. This means that in this figure either the boundary $f^i_{11:00}$, or the boundary $f^i_{10:01}$ has positive length. We elaborate on the two cases separately.

Case A: the boundary $f^i_{11:00}$, exists in the figure of player i.

Let $f^i_{11:00} = f^i_{a^{ii}:a^{jk}}$, where $j = k$ is allowed. We assume w.l.o.g. that either $j, k \neq 1$, or $j = 1$, (and possibly $k = 1$, too). Let $t_i = (t_{i1}, t_{i2})$ and $t'_i = (t_{i1}, t_{i2} + \delta)$, so that $t_i \in R^i_{11}$ and $t'_i \in R^i_{00}$. We consider the allocation figures of player 1/ [1] for t_i and for t'_i. Since the figure of player 1 is quasi-independent except for maybe tie-breaks, we can perturb all bids by a small value, including t_i, so that in both cases the figure of 1 is quasi-independent, and t_1 is not a boundary point. Assume that $j, k \neq 1$. In that case, t_1 gets from a^{ii} to a^{jk} as t_i is changed to t'_i, so the allocation in R_{00} of player 1 is changed, and since the figures are quasi-independent, also the allocation in R_{01} must be changed from a^{i1} to a^{j1}. However, this is not possible due to a simple increase in t_{i2}, by SMON. Now assume that $j = 1$. In this case, due to the increase of t_{i2}, the bid t_1 gets from region R_{00} into R_{10} or R_{11}, also a contradiction, since only the horizontal boundary (if any) can increase, as we noted above.

Case B: the boundary $f^i_{10:01}$, exists in the figure of player i.

Let $f^i_{10:01} = f^i_{a^{ij}:a^{ki}}$, where $j = k$ is allowed. If $j \neq 1$, then t_1 gets from R_{00} into R_{00} or R_{10} due to *decreasing* t_{i2}, and the proof is completely analogous to the previous case. The same holds if $k \neq 1$. Finally, if $f^i_{10:01} = f^i_{a^{i1}:a^{1i}}$, then t_1 gets from R_{10} to R_{01} if t_{i2} increases, again impossible by changing the horizontal boundary.

Proof of Lemma 4. We show that all allocation figures of player 1 are crossing, and this will imply the theorem by Lemma 3.

Assume first, that for some t_{-1} with $t_{i1} = \bar{t}_{i1}$, the figure of player 1 has the boundary $f_{a^{11}:a^{i1}}$, but is non-crossing. We show that then there exists another t''_{-1} with the same properties, that even has an open neighborhood $\mathcal{G}$ with constant allocations in all regions of 1, contradicting Theorem 2. Let us fix four different bids $t^1_1, t^2_1, t^3_1, t^4_1$ of player 1 in the interiors of the regions R_{11}, R_{01}, R_{10}, and R_{00}, respectively. We can add an arbitrarily small positive vector to each bid t_j $(j \neq 1, i)$ one by one, so that for each of $t^1_1, t^2_1, t^3_1, t^4_1$, the t_j is not on any boundary in the figure of j, moreover the allocations a^{11} and a^{i1} are still valid in the figure of 1 (since we increased the other bids). If now t_i is on some boundary for any of the fixed t_1 values, then we further increase a t_j bid that this boundary depends on (this cannot be t_1, since the four t_1 are internal points). Now we found a t'_{-1} where every value is an internal point of its region for all four t_1 values, which means that the allocations do not change in some open neighborhood $\mathcal{G}$ of t'_{-1}. Consequently, Theorem 2 applies, and for t'_{-1} we have a crossing figure of player 1. Since we changed the bids by arbitrarily small vectors, which moved the boundaries in the figure of 1 by arbitrarily little, therefore the original figure must have been crossing as well.

Next, we fix a t_{-1} with $t_{i1} = \bar{t}_{i1}$, so that the figure of 1 is crossing, and even in some open neighborhood $\mathcal{G}$ of t_{-1}, the allocations in the figure of 1 do not change. For this t_{-1}

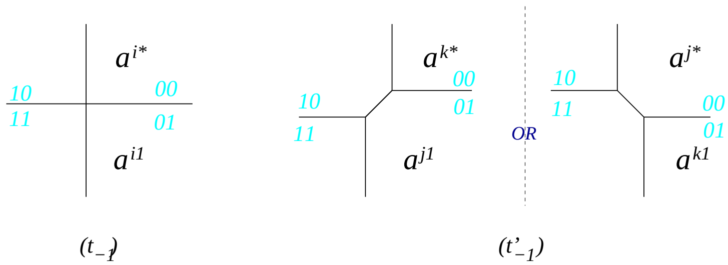

Figure 5: Proof of Lemma 4.

the allocation must be quasi-independent, otherwise, by analogous argument as in Lemma 3, the figure can be made non-crossing, contradicting the previous paragraph. Now, assume for contradiction that for some arbitrary fixed other bids t'_{-1} the figure of 1 is non-crossing, that is, either $f_{11:01} < f_{10:00}$, or $f_{10:00} < f_{11:01}$ holds in the figure of 1. Let j denote the player who receives task 1 in R_{01} if the first case holds, resp. in R_{00} if the second case holds, and let k be the player who receives task 1 in the other region (R_{00} or R_{01}, respectively), where possibly $k = j$ (see Figure 5). As the first step, we increase every $t_{\ell 1}$ *and* $t'_{\ell 1}$ bid for $\ell \neq i, j$ to some common high value K in *both* t_{-1} and t'_{-1}. Here K is higher than any bid in t_{-1} or t'_{-1}. This does not change the figure for t_{-1}, since neither the F, nor the G value can change, and the figure must remain crossing according to the previous paragraph. Observe that, because of increasing t_{k1}, it might change the figure for t'_{-1}, but only by distorting its shape even further away from a crossing figure. As a second step, we also set $t'_{i1} = K$ in t'_{-1} (unless $j = i$), without changing the figure for t'_{-1}.

In what follows, we modify the bids in t_{-1} further, to finally obtain t'_{-1}, and show that the crossing figure cannot become non-crossing during this modification. Next, we set the value $t_{\ell 2}$ to $t'_{\ell 2}$ for all $\ell \neq 1$. Notice that this does not influence the existence and position of boundary $f_{a^{11}:a^{i1}}$ (it can change only the horizontal boundary and the other allocations). It remains to set t_{j1} and t_{i1} to t'_{j1} and t'_{i1}, respectively. If $i \neq j$, we decrease t_{j1} until either in R_{00} or in R_{01} player j receives the first task. Note that for some t_{j1} this must occur, because of the figure of player j is complete (for some fixed t_1), and t_j arrives at the region R^j_{10} or R^j_{11} at some point. We claim that as we decrease t_{j1}, player j takes over the first task in both of these regions at exactly the same t_{j1} value, and then the vertical boundary at $f_{11:01}$ starts to move to the left from its original position (at $F(\bar{t}_{i1})$). Were this not the case, the figure would 'immediately' become non-crossing, while either the boundary $f_{11:01}$ or the boundary $f_{10:00}$ still depends on t_{i1} by the locally nonlinear function F. This cannot be the case for $f_{11:01}$ by Theorem 2 but also not for $f_{10:00}$, since in this case our locally nonlinear F function would correspond to one of the H functions in cases (4), (5), or (6) (here we exploit that the allocation was originally quasi-independent, so the same player receives task 2 in R_{10} and in R_{00}; the roles of F and G are exchanged in the figure). In fact, as player j takes over the first task, the allocation must be again quasi-independent, and the figure remains crossing. Thus, we can go on decreasing t_j1 to a very low value, while we set t_{i1} to t'_{i1} without changing the allocations and the figure (t_{j1} is low enough so that player j receives the first task for both t_{i1} and t'_{i1}). Finally, we increase t_{j1} back to t'_{j1}. Now the bids of the players other than 1 are set as in t'_{-1}, so the figure of 1 is non-crossing by assumption. It is easy to check, that even if during this increase j loses the first task in either R_{00} or R_{01} against some other player k, the figure gets distorted into the assumed shape, but with flipped roles of j and k, a contradiction (see Figure fig:threshold). If j does not lose a task (or, say $j = k$), then the figure remains crossing, again contradicting the assumption.

Finally, assume that $i = j$. Then $t_{i1} < t'_{i1}$ must hold, otherwise t_{i1} could be decreased to t_{i1} maintaining local independence and the crossing figure. Now we can increase t_{i1} to t'_{i1}, and like above, the figure cannot take on the assumed shape. Notice that the proof remains valid also if $i = k$.

Proof of Theorem 4. (a) If F_i were locally non-linear in some internal point of its domain, then the mechanism would be task-independent by Lemma 4. Therefore, F_i is locally linear in every internal point of its domain. Hence, F_i is linear on any closed bounded sub-interval $[-K, c-\delta]$ of its domain, since the interval is compact, and can be covered by finitely many open neighborhoods on each of which the function is linear. Finally, by taking $K \to \infty$ and $\delta \to 0$, linearity extends to the whole domain $(-\infty, c)$.

(b) Let the allocation a^{j1} be valid in R_{01}, implying $f_{11:01} = F_j(t_{j1})$. Assume for contradiction that $F_i(t_{i1}) < F_j(t_{j1})$ for some other player $i \neq j$. Since the function F_i is defined, the allocation a^{i1} does exist for some t'_{-1}. We change the bids in t_{-1} and t'_{-1} one by one, to make them equal. First, we increase every t_{k1} and t'_{k1} for $k \neq i, j$, to a common high value (with very high $F_k(t_{k1})$), thereby not changing the allocation in R_{01} for t_{-1}, or for t'_{-1}. Furthermore, a^{i1} remains the allocation for t'_{-1}, if we decrease t'_{i1} below t_{i1}, and increase t'_{j1} above t_{j1} when necessary. Next, we change t_{-1} by increasing t_{j1} to t'_{j1}. If this changes the allocation a^{j1} to a^{k1} or a^{i1}, then the boundary $f_{11:01}$ must have jumped up to $F_k(t_{k1}) > F_j(t'_{j1})$, or jumped down to $F_i(t_{i1}) < F_j(t_{j1})$, respectively, contradicting continuity. Finally, moving t_{i1} down to t'_{i1} changes the allocation to a^{i1} (since by now $t_{-1} = t'_{-1}$) and implies a jump of the boundary to $F_i(t_{i1}) < F_j(t_{j1})$, a contradiction.

(c) By analogous argument as in (b) it follows that the domain of any F_i (when defined), cannot 'stop' at some finite upper bound c. Finally, (d) follows directly from (a) and (b).

Proof of Theorem 5. We assume that in a neighborhood of some fixed t_{-1}, the boundary function $f_{10:00}$ has an additive component $H(t_{i1}+t_{i2})$. Then for this set of t_{-1} the allocation a^{ii} is valid in the region R_{00}. We show that H is linear with the given slope. According to Theorem 4 (d), the values t_{j1} for $j \neq 1, i$ can be increased so that in R_{01} the allocation a^{i1} appears. Moreover, increasing these bids does not change the allocations in R_{11}, R_{10}, and in R_{00}, and does not change the boundary position $f_{10:00}$. Now by increasing $\bar{t}_{i1}$ by a small δ, the boundaries $f_{10:00}$ and $f_{11:01}$ move parallel by $\lambda^1_{i,horiz} \cdot \delta$, by the same slope as $F_i(t_{i1})$. However, if we increase $\bar{t}_{i1}$ *before* increasing all other t_{j1}, exactly the same increase is achieved, due to the function $H(t_{i1} + t_{i2})$. So, H must have

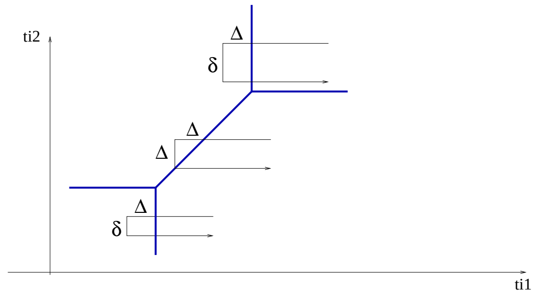

Figure 6: Illustration to Theorem 5. The three cases depend on whether the allocation of player i was 01 (in region R^1_{00}) or 00 (in region R^1_{01}), and if it became 11 or 10 (in region R^1_{00}).

slope $\lambda^1_{i,horiz}$ as well. Exactly the same argument remains valid, if H depends on t_{i1}, but not on t_{i2}, because R_{00} has the allocation a^{ik} for some $k \neq i$.

The case when H depends only on t_{i2} (and possibly on some t_{k1}), needs a more careful treatment. We decrease t_{i1} down to the point where task 1 is allocated to player i either in R_{00}, or in R_{01}, or both (one of these occurs for small enough t_{i1} by Theorem 4 (d)) Assume, e.g., that the allocation in R_{00}, becomes a^{ii}, so that t_i is now in region R^i_{11} for some fixed $t_1 \in R_{00}$ (see Figure 6). After the boundary point we further decrease t_{i1} by a small Δ, thereby decreasing the boundary $f_{10:00}$ by $\lambda^1_{i,horiz} \cdot \Delta$, according to the previous paragraph. Next, we decrease t_{i2} by δ, which further decreases this boundary by the same slope $\lambda^1_{i,horiz} \cdot \delta$. Finally, we increase t_{i1} by Δ to the boundary point of t_i, and then back to its original value, which increases the boundary by $\lambda^1_{i,horiz} \cdot \Delta$. Altogether, we decreased t_{i2} by δ, and this moved the boundary $f_{10:00}$ by $\lambda^1_{i,horiz} \cdot \delta$, verifying the declared slope of H as a function of t_{i2}. The case when either R_{00} or R_{01} becomes a^{ik}, can be handled by similar arguments, as shown in Figure 6. Note that until t_{i1} reached a non-horizontal boundary point somewhere, no allocation and no boundary changes at all.

Proof of Lemma 5. We assume w.l.o.g., that $k = 1$, and that the allocations a^{i1} and a^{1i} both exist. Since the mechanism is not task-independent, player 1 has a non-crossing allocation figure by Lemma 3.

Case 1. The non-crossing figure has a shape like in Figure 1 (a).

We decrease t_{i1} until in at least one of the regions R_{01} or R_{00} player i receives task 1 (this happens for some t_{i1} by Theorem 4 (d)). Moreover, *before* it happens, none of the four allocations change, and the figure has constant shape. If player i receives task 1 in region R_{00} *strictly before* the same happens in region R_{01}, then for an interval of t_{i1} values the allocation is of type (4)–(9), and Corollary 2 (also using Observation 5) implies $\lambda^k_{i,horiz} = \lambda^k_{i,vert}$. Otherwise, if playes i gets task 1 in R_{01} before or at the same time as in R_{00}, then R_{01} has allocation a^{i1} in a non-crossing figure of still the same shape.

Now similarly we decrease t_{i2} until player i receives task 2 in R_{00} or in R_{10} or both. By the same argument, we can assume that in R_{10} the allocation is a^{1i}, the figure is non-crossing, because of the shape the slanted boundary $f_{10:01}$ exists. However, now this boundary changes by a function of $(t_{i1} - t_{i2})$ due to $(\star)$. At the same time, the boundary position is a linear function of t_{i1} with slope $\lambda^k_{i,horiz}$ (together with $f_{11:01}$), and a linear function of $-t_{i2}$ with slope $\lambda^k_{i,vert}$ (together with $f_{11:10}$). This is possible only if $\lambda^k_{i,horiz} = \lambda^k_{i,vert}$.

Case 2. The non-crossing figure has a shape like in Figure 1 (c).

The proof is analogous to that of Case 1 up to the point where R_{10} and R_{01} have the allocations a^{1i} and a^{i1}, and R_{00} (possibly) not. Now we decrease t_{i1} further, until either in R_{00} player i gets task 1 (or both tasks), or the figure changes its shape and Case 1. can be applied. But what if player i 'takes over' task 1 (both tasks in fact) in R_{00} at the very point where the figure becomes crossing (and the same happens when we decrease t_{i2})? In this case it can be verified that for any (large enough) fixed $t_1 \in R_{00}$ point, in the figure of player i a slanted boundary of the form $\lambda^k_{i,horiz} t_{i1} + \lambda^k_{i,vert} t_{i2} = c$ appears, implying $\lambda^k_{i,horiz} = \lambda^k_{i,vert}$.

Proof of Lemma 2.3. Let $k = 1$. If $1, i, j$ is a triangle, then it can be assumed (modulo permutation of the player indices) that the allocations a^{1j} and a^{i1} exist. Since the mechanism is not task-independent, player 1 has a non crossing figure. We start from the non-crossing allocation figure of player 1 and decrease t_{i1} until in one of the regions R_{01} or R_{00} player i receives task 1. Similarly, we decrease t_{j2} until in one of the regions R_{10} or R_{00} player j receives task 2. The proof is now similar to the proof of Lemma 5: in any case we produce a figure of type (2)–(9), and the equality follows from Lemma 2 (c).

Proof of Theorem 6. Consider first neighboring nodes i and k, with the above defined λ^k_i. For F^k_i and G^k_i the statement follows from the definition of λ^k_i and from Lemma 5; for H it holds by Theorem 5. For other players i connected to k by a longer path we prove the theorem by induction on the distance of k and i. Note that for such players F^k_i and G^k_i are undefined because a^{ik} and a^{ki} never occur. We need to treat only those cases when in R^k_{00} the allocation is a^{ii} or a^{ij}, and the position of boundaries of R^k_{00} have an additive component $H(t_{i1}+t_{i2})$ or $H(t_{i1}, t_{j2})$. We consider first a fixed allocation figure with a^{ii} in R^k_{00}

Let w.l.o.g. $k = 1$, and let $1, \ell \ldots, i$ be a shortest path between 1 and i. Assume w.l.o.g. that $a^{\ell 1}$ exists. According to Theorem 4 (d) the t_{s1} Werte of all players $s \neq 1, i, \ell$ can be increased so that in R_{01} the allocation becomes $a^{\ell 1}$ (t_{i1} needs not be increased, since a^{i1} does not occur for any bid). Due to increasing these t_{s1} bids, the allocations in R_{10} and in R_{00} do not change, neither changes the boundary $f_{10:00}$ between them. Now if the allocation in R_{01} is $a^{\ell 1}$, then for the t_1 points of the boundary $f_{01:00}$ it holds that $t_{12} = H(t_{i1} + t_{i2}) - \lambda^1_\ell t_{\ell 1} + c$ where c is a constant. Now looking at the same (inverse) boundary in the allocation figure of player ℓ, there it holds that $t_{\ell 1} = \lambda^\ell_i (t_{i1} + t_{i2}) - \lambda^\ell_1 t_{12} + d$. Here we exploit the induction hypothesis: since the distance of ℓ and i is smaller than the distance of 1 and i in the player graph, the λ_i^ℓ has been defined and the dependence of $f^\ell_{10:00}$ on t_i is linear with this slope. Comparing the equivalent equalities yields that H is linear (with slope $\lambda_i^\ell/\lambda_1^\ell = \lambda_\ell^1/\lambda_\ell^i$). Recall that we started from an arbitrary figure of 1 where a^{ii} appeared, and the choice of ℓ was independent of the figure and of t_{-1}. We define the constant slope of H to be λ_i^1. The case when in R_{00} the allocation is a^{ij}, can be treated the same way, by induction on the distance of the edge $\{i,j\}$ from vertex k in the player-graph.

Proof of Observation 3. For simplicity of notation we assume that $(i_0, i_1, i_2, \ldots, i_t) = (0, 1, \ldots, t)$, and prove the statement by induction on t. For length $t = 1$ it is trivial. Assume that it holds for paths of length $t-1$. If $\{0,t\}$ is not and edge, then the argument is the same as in the proof of Theorem 6 (there we did not use that the path was a shortest path, just that $\{1,i\}$ is not an edge, that is, a^{i1} does not exist).

If $\{0,t\}$ is an edge, then we claim that $\lambda_0^{t-1} = \lambda_t^{t-1} \cdot \lambda_0^t$. This will conclude the proof by rearranging to $\lambda_{t-1}^t \cdot \lambda_0^{t-1} = \lambda_0^t$, and applying the induction hypothesis. If $\{0, t-1\}$ is also an edge, then the claim holds by Lemma 2.3. If $\{0, t-1\}$ is not an edge, then it holds by the same proof as in Theorem 6, applied for the path $0, t, t-1$.

Proof of Corollary 3. The claim follows by taking (shortest) paths from i to j and from j to k. If these paths have common edges from j to some player, the corresponding λ values cancel out and can be omitted from the path from i to k.

Proof of Lemma 7. For simplicity, we sketch the proof for the case when each λ_i^k equals 1. The c^{ij} values can be defined in a straightforward way, and the consistency of these definitions (i.e. validity in all allocation figures) follows easily. W.l.o.g. we define $c^{11} = 0$. If an allocation a^{ij} never occurs, then we set $c^{ij} = \infty$. Otherwise the c^{i1} values are the additive constants in $F_i(t_{i1})$ (since $\lambda_i^1 = 1$), and similarly for c^{1i}. For every other allocation a^{ii} or a^{ij}, consider any figure of player 1 with this allocation in R_{00}. If the boundary $f_{11:00}$ does not appear in the figure, then we define a *virtual boundary* of the form $f_{11:00} = t_{11} + t_{12}$ to always mean the symmetry axis between regions R_{11} and R_{00}. The c^{ij} is determined by the position of the (virtual) boundary $f_{11:00}$, which is a linear function of t_{i1} and t_{j2} (of slope 1 for both, by assumption). Let c^{ij} be such that $f_{11:00} = t_{i1} + t_{j2} + c^{ij}$.

Why are these definitions consistent? Consider for example a boundary $f^i_{a^{ii}:a^{ji}}$ in the figure of player i. We have to show that this boundary has the equation $t_{i1} = t_{j1} + c^{ji} - c^{ii}$ given the above definition of c^{ji} and c^{ii}. However, if in the figure of player 1 in R_{00} the allocation is a^{ji}, and we decrease t_{i1}, the boundary $f_{11:00}$ of R_{00} does not change until the t_{i1} reaches the boundary ($t_{i1} = f^i_{a^{ii}:a^{ji}}$), where the allocation changes to a^{ii}. At this very point both $f_{11:00} = t_{j1} + t_{i2} + c^{ji}$ and $f_{11:00} = t_{i1} + t_{i2} + c^{ii}$ hold. Putting it all together $f^i_{a^{ii}:a^{ji}} = t_{j1} + c^{ji} - c^{ii}$ which concludes the proof of this case.

Proof of Lemma 8. The proof follows the same lines as the definition and consistency proof of c^{ii} values. Assuming that player k is the representative of any particular group g, we define C_g to be the supremum of $(t_{k1} + t_{k2})$ values such that with this t_k player k ever gets both tasks. It is easy to see that it does not depend on the representative whether C_g is finite or infinite for a given group. Moreover, for at least one group $C_g = \infty$ must hold, otherwise for 'very high' bids no allocation would be determined. Assume w.l.o.g. that $C_1 = \infty$. We define the Φ_g functions relative to group 1 (to player 1, in fact) and fix Φ_1 to be the identity function.

Let $k = k^g$ be the representative of a fixed group g. Recall that $c^{kk} = 0$ because we normalized to zero the additive constants for a^{kk} of group-representatives. For arbitrary value $x < C_g$ we determine $\Phi_g(x)$ as follows. The mechanism allocates a^{kk} for some t such that $t_{k1} + t_{k2} > x$ (by definition of C_g), and by monotonicity the allocation remains if we decrease t_{k1} or t_{k2} so that $t_{k1} + t_{k2} = x$ now the position of the (virtual) boundary $f_{11:00}$ in the figure of player 1 determines $\Phi_g(x)$, that is $\Phi_g(x) = \Phi_g(t_{k1} + t_{k2}) = t_{11} + t_{12}$ should hold (in fact, the bids in group 1 can always be increased so high that the boundary $f_{11:00}$ really appears).

Checking the consistency of these definitions means checking that in the figure of a player i of group g the region R^i_{00} has boundaries determined by $H = \min_h \Phi_g^{-1}(\Phi_h(Opt_h))$. This can be done analogously to checking the consistency of c^{ij} in Lemma 7.

Since the inverse function Φ_g^{-1} would be Φ_1 if we exchanged the roles of 1 and g, it follows that $\lim_{-\infty} \Phi_g = -\infty$ must hold. Furthermore, continuity requires that if C_g is finite then $\lim_{C_g} \Phi_g = \infty$ (we can define $\Phi_g(x) = \infty$ for $x \geq C_g$).